Introduction to Econometrics (Part. III)

Giovanni Angelini

Rimini, 9 September 2013 - 19 September 2013

Outline

- Time series analysis
- Model
- OLS estimation
- Testing linear hypotheses

- Henceforth we use the index t instead of i to denote ‘time’.
- Let y_t be a scalar (stochastic) variable of interest. **Endogenous variable.**
- Let x_t a $f \times 1$ vector of (stochastic) **contemporaneous explanatory variables**. These are variables that, according to out theory of view of the phenomenon under investigation, have a direct impact on y_t.
- The variables y_t and x_t are observed at time $t = 1, 2, ..., T$, so the available data (time series) are:

$$y_1, y_2, ..., y_T$$
$$x_{1,} x_2, ..., x_T.$$

Model for 'Investment' in USA

- Dataset:
    - i: investment (real investment)
    - y:GDP
    - R: long term interest rate
    - ff: funds rate
    - π: inflation rate
- Expecially in macroeconomics the variables are non stationary. Is necessary to trasform the nonstationary variables in stationary variables before estimation.

- Example:

$$\triangle\, i_t = \beta_0 + \beta_1 \triangle\, i_{t-1} + \beta_2 \triangle\, y_t + \beta_3 \triangle\, y_{t-1} + \beta_4 \left(ff_{t-1} - \pi_{t-1} \right) + u_t$$

- Can we see the non-stationary graphically?
  1. Investment
  2. First Difference on Investment
  3. GDP
  4. First Difference on GDP
  5. Federal Funds Rate - Inflation

- Formally, the process X_t is Covariance Stationary if:

$$\begin{aligned} E\left(X_t\right) &= \mu_X < \infty \\ Cov(X_t, X_{t-k}) &= E\left[\left(X_t - \mu_X\right)\left(X_{t-k} - \mu_X\right)'\right] \\ &\quad \text{does not depend on t} \end{aligned}$$

- Now we can assume that all the variables in our model are stationary, we have the relationship of the form:

$$\triangle\, i_t = \beta_0 + \beta_1 \triangle\, i_{t-1} + \beta_2 \triangle\, y_t + \beta_3 \triangle\, y_{t-1} + \beta_4 \left(ff_{t-1} - \pi_{t-1} \right) + u_t$$

- Where:
  - $\beta_i, i = 0, 1, 2, 3, 4$ are the parameters of interest
  - u_t is a stochastic disturbance term

- u_t is a (stochastic) disturbance term with the following properties:
  - $E(u_t) = 0$ for each t (non-systematic disturbance);
  - $Cov(u_t, u_{t-h}) = 0$ for $h = 1, 2, ...$ (**serial uncorrelation**);
  - $Var(u_t) = E(u_t^2) = \sigma_u^2$ for eact t (**homoskedasticity**).
- The term u_t is called a scalar, $u_t \sim$WN(0,σ_u^2).
- The model will be denoted as **dynamic linear regression model.**
- Also known as **ADL** model, where A=Autoregressive, D=distributed L=lags.

Estimation

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.316714	0.139916	-2.263611	0.0246
DI(-1)	0.222151	0.059748	3.718132	0.0003
DY	1.438749	0.118686	12.12236	0.0000
DY(-1)	0.490749	0.153296	3.201323	0.0016
FF_INF(-1)	-0.446921	0.157793	-2.832328	0.0050

R-squared	0.632582	Mean dependent var	0.335439
Adjusted R-squared	0.626021	S.D. dependent var	2.453241
S.E. of regression	1.500251	Akaike info criterion	3.670734
Sum squared resid	504.1687	Schwarz criterion	3.745706
Log likelihood	-415.2990	Hannan-Quinn criter.	3.700980
F-statistic	96.41492	Durbin-Watson stat	2.067095
Prob(F-statistic)	0.000000		

Dynamic multiplier

- Consider as an example, the model

$$y_t = \beta_0 + \beta_1 y_{t-1} + \delta z_t + u_t$$

where z_t is a scalar.

Question 1: which is the instantaneous impact of z on y ?
Answer: it is $\delta := \frac{\partial y_t}{\partial z_t}$.
Question 2: which is the impact of impact of z on y after one period ?. Answer: you are looking for

$$\frac{\partial y_{t+1}}{\partial z_t} \quad \text{impact multiplier after one period.}$$

- Now

$$y_{t+1} = \beta_0 + \beta_1 y_t + \delta\ z_{t+1} + u_{t+1}$$

hence

$$\frac{\partial y_{t+1}}{\partial z_t} := \beta_1 \frac{\partial y_t}{\partial z_t} + \delta\ \frac{\partial z_{t+1}}{\partial z_t}$$

$$:= \beta_1 \delta + \delta\ \frac{\partial z_{t+1}}{\partial z_t}.$$

Assuming that $\frac{\partial z_{t+1}}{\partial z_t} := c = \text{const}$,

$$\frac{\partial y_{t+1}}{\partial z_t} := \delta(\beta_1 + c) \quad \text{impact multiplier after one period.}$$

- In general, we can compute

$$\frac{\partial y_{t+h}}{\partial z_t} \quad \text{impact multiplier after } h \text{ periods.}$$

Long run multiplier

- **Question 3:** which is the long run impact of z on y ?
  Answer: you are looking for:

  $$\frac{\partial E(y_t)}{\partial E(z_t)} \text{ long run multiplier.}$$

  Applying the expectations operator to both sides of

  $$y_t = \beta_0 + \beta_1 y_{t-1} + \delta\ z_t + u_t$$

  gives

  $$E(y_t) = \beta_0 + \beta_1 E(y_{t-1}) + \delta E(z_t).$$

- Because of the stationarity assumption $E(y_{t-1}) \equiv E(y_t)$, hence:

$$E(y_t) = \frac{\beta_0}{(1-\beta_1)} + \frac{\delta}{(1-\beta_1)}\ E(z_t)$$

which implies

$$\frac{\partial E(y_t)}{\partial E(z_t)} := \frac{\delta}{(1-\beta_1)} \quad \text{long run multiplier.}$$

- We can compact the ADL model in a more familiar form:

$$y_t = \beta' x_t + u_t \quad , \quad u_t \sim WN(0, \sigma_u^2) \quad , \quad t = 1, ..., T$$

where:

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \quad , \quad x_t = \begin{pmatrix} 1 \\ \triangle\, i_{t-1} \\ \triangle\, y_t \\ \triangle\, y_{t-1} \\ ff_{t-1} - \pi_{t-1} \end{pmatrix} \text{ are } 5 \times 1$$

- The unknown parameters are $\theta = (\beta', \sigma_u^2)'$,

- Our intrepretation of the dynamic regression model reamins the same:

$$y_t = E(y_t \mid x_t) + u_t \quad , \quad t = 1, ..., T$$

where

$$E(y_t \mid x_t) := \beta' x_t .$$

- In other words, we **condition** our explanation of the endogenous variable y_t with respect to a set of stochastic variables x_t and, at the same time, **assume that** $E(y_t \mid x_t)$ **be linear** !

Compact Representation

- Compact - Matrix - Representation

$$\underset{T\times 1}{y} = \underset{T\times k}{X}\,\underset{k\times 1}{\beta} + \underset{T\times 1}{u}, \quad E(uu') = \sigma_u^2 I_T$$

- Vector of unknown parameters: $\theta = (\beta', \sigma_u^2)'$.

OLS

- Objective function:

$$Q(\theta) = \frac{1}{\sigma_u^2} \sum_{t=1}^{T} (y_t - x_t' \beta)^2 \equiv \frac{1}{\sigma_u^2} (y - X \beta)'(y - X \beta).$$

- OLS estimator of β is obtained by solving the problem

$$\min_{\beta} Q(\theta)$$

i.e. the OLS estimator of β is the vector that solves:

$$\hat{\beta}_{OLS} = \arg\min_{\beta} Q(\theta).$$

- Solution to first order conditions leads us to

$$\hat{\beta}_{OLS} = \left(\sum_{t=1}^{T} x_t x_t' \right)^{-1} \left(\sum_{t=1}^{T} x_t y_t \right) \equiv \left(X'X \right)^{-1} \left(X'Y \right)$$

whereas the estimator of σ_u^2 is obtained indirectly

$$\hat{\sigma}_u^2 = \frac{1}{T-k} \left(\sum_{t=1}^{T} \hat{u}_t^2 \right) = \frac{1}{T-k} \hat{u}' \hat{u}$$

where $\hat{u}_t = y_t - x_t' \hat{\beta}$, $t = 1, ..., T$ or, alternatively, $\hat{u} = (y - X \ \hat{\beta})$.

- Are the estimators of β and σ_u^2 correct ?
  $E(\hat{\beta}_{OLS})=\beta$, $E(\hat{\sigma}_u^2)=\sigma_u^2$?? (if yes, under which conditions ?)
- Consider that

$$\hat{\beta}_{OLS} := (X'X)^{-1}X'[X\beta + u] = \beta + (X'X)^{-1}X'u$$

- and

$$E(\hat{\beta}_{OLS}) := E_X\left(E\left(\hat{\beta}_{OLS} \mid X\right)\right).$$

- Likewise, $E(\hat{\sigma}_u^2) := E_X\left(E\left(\hat{\sigma}_u^2 \mid X\right)\right)$.

- Estimator of β.

$$E\left(\hat{\beta}_{OLS} \mid X\right) = \beta + (X'X)^{-1}X'E\left(u \mid X\right).$$

Hence, if $E\left(u \mid X\right) = 0_{n\times 1}$, one has

$$E\left(\hat{\beta}_{OLS} \mid X\right) = \beta$$

$$\Rightarrow E(\hat{\beta}_{OLS}) := E_X\left(E\left(\hat{\beta}_{OLS} \mid X\right)\right) = E_X\left(\beta\right) = \beta,$$

the OLS estimator is correct **(for cross-section data)**.

- Note that for Time-Series data $E\left(u \mid X\right) \neq 0_{n\times 1}$ **hence the estimator is no longer correct** (it happens in the regression model with time series data in which regressors include lags of y).

Asymptotic properties (large T) of OLS estimator

H1: correct specification All underlying assumptions are fulfilled.

H2: stationarity and ergodicity The stochastic process that generates $w_t = (y_t, z_t')'$is covariance stationary and ergodic.

Then

P1: Consistency

$$\hat{\beta}_{OLS} \rightarrow_p \beta$$

P2: Asymptotic Normality

$$T^{1/2}\left(\hat{\beta}_{OLS} - \beta\right) \rightarrow_D N(0_{k \times 1}, \sigma_u^2 \Sigma_{xx}^{-1})$$

where $\Sigma_{xx} = \underset{k \times k}{E(x_t x_t')}$.

- Observe that

$$\hat{u}_t = (y_t - x_t \hat{\beta}'_{OLS}) \rightarrow_p (y_t - x_t \beta') = u_t \quad , \quad t = 1, ..., T$$

hence

$$\hat{u}_t^2 \rightarrow_p u_t^2 \qquad , \quad t = 1, ..., T$$

which implies that

$$E(\hat{u}_t^2) \approx E(u_t^2) \quad \text{for large } T.$$

- Therefore, under H1 and H2:

$$\hat{\sigma}_u^2 = \frac{1}{T-k} \left(\sum_{t=1}^{T} \hat{u}_t^2 \right) \rightarrow_p E(\hat{u}_t^2) = E(u_t^2) = \sigma_u^2$$

that means that if $\hat{\beta}_{OLS}$ is consistent for β **also $\hat{\sigma}_u^2$ is consistent for σ_u^2 !**

- Which is the variance of $\hat{\beta}_{OLS}$?

$$
\begin{aligned}
Var\left(\hat{\beta}_{OLS} \mid X\right) &= Var\left(\beta + (X'X)^{-1}X'u \mid X\right) \\
&= Var\left((X'X)^{-1}X'u \mid X\right) \\
&= (X'X)^{-1}X'\left(\sigma_u^2 I_n\right)X(X'X)^{-1} \\
&= \sigma_u^2 (X'X)^{-1}.
\end{aligned}
$$

- Why do we care about the covariance matrix of the estimator $\hat{\beta}_{OLS}$?
- Look again at the output:

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Hypothesis Testing

- Because it is fundamental to make inference !

$$t - statistic = \frac{coefficient}{s.e.(coefficient)}$$

- In this way we can test if the coefficient is statistically different from 0.
- Because it can be proof that t-statistic is distribuited like a $t(t-k)$. T-distribution is approximatively a $N(0,1)$ distribution for large sample.
- Comparing the value obtained for the t-statistic and the asymptotic distribution we can decide if a coefficient is statistically different from 0.